

Interacting multiple zero mode formulation and its application to a system consisting of a dark soliton in a condensate

J. Takahashi,^{1,*} Y. Nakamura,^{1,†} and Y. Yamanaka^{1,‡}

¹*Department of Electronic and Physical Systems, Waseda University, Tokyo 169-8555, Japan*
(Dated: April 27, 2015)

To formulate the zero modes in a finite-size system with spontaneous breakdown of symmetries in quantum field theory is not trivial, for in the naive Bogoliubov theory, one encounters difficulties such as phase diffusion, the absence of a definite criterion for determining the ground state, and infrared divergences. A new interacting zero mode formulation that has been proposed for systems with a single zero mode to avoid these difficulties is extended to general systems with multiple zero modes. It naturally and definitely gives the interactions among the quantized zero modes, the consequences of which can be observed experimentally. In this paper, as a typical example, we consider an atomic Bose–Einstein condensed system with a dark soliton that contains two zero modes corresponding to spontaneous breakdown of the U(1) gauge and translational symmetries. Then we evaluate the standard deviations of the zero mode operators and see how the mutual interaction between the two zero modes affects them.

PACS numbers: 03.75.Hh, 03.75.Nt, 67.85.-d

I. INTRODUCTION

Since the pioneering experiments [1–3], the Bose–Einstein condensate (BEC) phenomenon in a trapped ultracold atomic system has been the central subject of many experimental and theoretical studies. We view the subject from the standpoint of quantum field theory, which is the most fundamental dynamical law and in which the BEC in an atomic system is interpreted as a spontaneous breakdown of the U(1) gauge symmetry.

The concept of spontaneous symmetry breaking (SSB) yields many examples of successful descriptions of nature using quantum field theory. In an infinite system with a spontaneously broken symmetry, the celebrated Nambu–Goldstone theorem [4, 5] implies the existence of a zero mode, reflecting the original symmetry. The zero mode plays a crucial role in creating and retaining the ordered state.

Study of the BEC in a trapped system highlights the importance of treating the operators belonging to the zero mode sector carefully because the system has a finite size, and the zero energy state also stands alone as a discrete level. The importance is easily overlooked in a homogeneous infinite system, for the zero mode sector is buried in the continuum labeled by the momentum index. The practice of formulation, usually called the Bogoliubov theory, is to take an unperturbed Hamiltonian up to the second power of a field operator and to attempt to diagonalize it by expanding the field operator in the appropriate complete set. However, when the complete set includes eigenfunctions belonging to a zero eigenvalue, *i.e.*, when the system has zero modes, the unperturbed Hamiltonian of the zero mode sector cannot be

diagonalized in terms of creation and annihilation operators but has the form of free particles expressed by the quantum mechanical operators \hat{P} and \hat{Q} . We refer to this as the *free zero mode formulation*. Although this part of the Hamiltonian is simply neglected for a homogeneous infinite system, it cannot be neglected for a finite-size system. Then, according to Ref. [6], the phase of the order parameter is definite only for a short time because \hat{Q} is interpreted as a phase operator, and its quantum fluctuation is given by $\Delta Q \sim t$ for large t . We also point out that there is no criterion for specifying a vacuum uniquely, as any energy eigenstate of a free particle has infinite ΔQ . Thus, it is concluded that the free zero mode formulation for a finite-size system is inconsistent.

To address this inconsistency, we proposed a new formulation for spontaneous breakdown of the U(1) gauge symmetry in Ref. [7]. The key point there is the inclusion of higher-than-third powers of \hat{P} and \hat{Q} in the unperturbed Hamiltonian, which yields their nonlinear equations of motion. We therefore call it the *interacting zero mode formulation*. The stationary Schrödinger-like equation with the nonlinear unperturbed Hamiltonian of \hat{P} and \hat{Q} gives bound states rather than the free one, and the energy spectrum becomes discrete, so the ground state is identified uniquely as the vacuum. Then ΔQ is independent of time, and we have no inconsistency as long as the calculated ΔQ is small.

The interacting zero mode formulation not only enables us to describe quantum fluctuation of a zero mode properly, but also, when two or more symmetries are broken spontaneously and there are multiple zero modes, introduces interactions among the zero modes naturally. In this paper, we focus on interactions among zero modes, extending the interacting zero mode formulation for a single zero mode [7] to that for multiple ones. After giving a general formulation, we consider, as an example of its application, a system consisting of a dark soliton in a homogeneous condensate, where two zero modes coexist

* j.takahashi@aoni.waseda.jp

† yusuke.n@asagi.waseda.jp

‡ yamanaka@waseda.jp

corresponding to the spontaneously broken translational and U(1) gauge symmetries. This example was studied in Ref. [8], in which a nonperturbative treatment of the zero modes using an effective Hamiltonian was proposed. However, the resultant Hamiltonian represents the free motion of each zero mode with no interaction, so our approach is essentially different from that in Ref. [8].

This paper is organized as follows: In Sect. II, we extend the interacting zero mode formulation for a single zero mode, presented in [7], to a general case of multiple zero modes, comparing it with the corresponding free zero mode case. In Sect. III, the general formulation is applied to the originally homogeneous system containing a condensate and a dark soliton. Two zero modes appear and interact with each other. We are particularly interested in the quantum fluctuations ΔQ_i (where i labels each zero mode), which are affected by interactions among the zero modes. Section IV presents a summary and conclusion.

II. FORMULATION OF MULTIPLE ZERO MODES IN QUANTUM FIELD THEORY

In Ref. [7], we proposed the interacting zero mode formulation for a finite-size system with spontaneous breakdown of the U(1) gauge symmetry. The main motivation for the formulation is that the quantum fluctuation in the phase of the order parameter cannot remain small in the conventional free zero mode formulation, whereas its smallness is the starting assumption of the formulation. In this section, we extend the new formulation to cases of multiple zero modes.

We suppose a Hamiltonian with global U(1) gauge symmetry,

$$\hat{H} = \int d^3x \left[\hat{\psi}^\dagger \left(-\frac{\nabla^2}{2m} + V_{\text{ex}} - \mu \right) \hat{\psi} + \frac{g}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right], \quad (1)$$

where V_{ex} , m , μ , and g are the external potential, atomic mass, chemical potential, and repulsive coupling constant ($g > 0$), respectively. Throughout this paper, we set \hbar to unity. The bosonic field operator $\hat{\psi}$ obeys the canonical commutation relations $[\hat{\psi}(\mathbf{x}, t), \hat{\psi}^\dagger(\mathbf{x}', t)] = \delta(\mathbf{x} - \mathbf{x}')$, $[\hat{\psi}(\mathbf{x}, t), \hat{\psi}(\mathbf{x}', t)] = 0$. When the U(1) gauge symmetry is broken spontaneously, $\hat{\psi}$ is divided into coherent and incoherent parts as $\hat{\psi} = \xi + \hat{\varphi}$, according to the criterion $\langle 0 | \hat{\varphi} | 0 \rangle = 0$. The coherent part, or the order parameter ξ , is related to the total number of condensates, $N_0 = \int d^3x |\xi|^2$, and the vacuum $|0\rangle$ is determined self-consistently later. The total Hamiltonian is rewritten

in terms of $\hat{\varphi}$ as $\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4$, where

$$\hat{H}_1 = \int d^3x \left[\hat{\varphi}^\dagger \left(-\frac{\nabla^2}{2m} + V_{\text{ex}} - \mu + g|\xi|^2 \right) \xi \right] + \text{h.c.}, \quad (2)$$

$$\hat{H}_2 = \int d^3x \left[\hat{\varphi}^\dagger \mathcal{L} \hat{\varphi} + \frac{1}{2} \hat{\varphi} \mathcal{M} \hat{\varphi} + \frac{1}{2} \hat{\varphi}^\dagger \mathcal{M}^* \hat{\varphi}^\dagger \right], \quad (3)$$

$$\hat{H}_3 = g \int d^3x \xi \hat{\varphi}^\dagger \hat{\varphi}^\dagger \hat{\varphi} + \text{h.c.}, \quad (4)$$

$$\hat{H}_4 = \frac{g}{2} \int d^3x \hat{\varphi}^\dagger \hat{\varphi}^\dagger \hat{\varphi} \hat{\varphi}, \quad (5)$$

where $\mathcal{L} = -\nabla^2/2m + V_{\text{ex}} - \mu + 2g|\xi|^2$ and $\mathcal{M} = g\xi^2$.

A. FREE ZERO MODE FORMULATION

In the conventional approach, one chooses $\hat{H}_1 + \hat{H}_2$ as the unperturbed Hamiltonian assuming small $\hat{\varphi}$:

$$\hat{H}_0 = \hat{H}_1 + \hat{H}_2. \quad (6)$$

Because the vacuum of this Hamiltonian is time-independent, the field division criterion $\langle 0 | \hat{\varphi}(\mathbf{x}, t) | 0 \rangle = 0$ and the Heisenberg equation yield $\hat{H}_1 = 0$. This implies that ξ should satisfy the Gross-Pitaevskii (GP) equation [11],

$$\left(-\frac{\nabla^2}{2m} + V_{\text{ex}} - \mu + g|\xi|^2 \right) \xi = 0. \quad (7)$$

To diagonalize \hat{H}_2 [6, 9], we introduce the Bogoliubov-de Gennes (BdG) equation [12, 13] $T\mathbf{y}_n = \omega_n \mathbf{y}_n$ with the doublet notations

$$T = \begin{pmatrix} \mathcal{L} & \mathcal{M} \\ -\mathcal{M}^* & -\mathcal{L} \end{pmatrix}, \quad \mathbf{y}_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}. \quad (8)$$

Now, we restrict ourselves to cases where all the eigenvalues are real, which implies that the system is dynamically stable, in order to diagonalize the excited modes. The diagonalization of a system with complex modes (that is, a dynamically unstable system) is discussed in Ref. [10]. We consider that some symmetries in addition to the U(1) gauge symmetry are spontaneously broken and we need two or more eigenfunctions belonging to zero eigenvalue, *i.e.*, $T\mathbf{y}_{0,i} = 0$ where $\mathbf{y}_{0,i} = (f_i, -f_i^*)^t$, with a label i . For the sake of completeness, one has to introduce an adjoint function $\mathbf{y}_{-1,i} = (h_i, h_i^*)^t$ to each $\mathbf{y}_{0,i}$, which satisfies $T\mathbf{y}_{0,i} = I_i \mathbf{y}_{-1,i}$ and $2 \int d^3x h_i^* f_i = 1$, with normalization constants I_i . Let us expand $\hat{\varphi}(\mathbf{x}, t)$ by the BdG complete set as $\hat{\varphi}(\mathbf{x}, t) = \hat{\varphi}_0(\mathbf{x}, t) + \hat{\varphi}_{\text{ex}}(\mathbf{x}, t)$, where

$$\hat{\varphi}_0(\mathbf{x}, t) = \sum_{i=\text{z.m.}} \left[-i\hat{Q}_i(t)f_i(\mathbf{x}) + \hat{P}_i(t)h_i(\mathbf{x}) \right], \quad (9)$$

$$\hat{\varphi}_{\text{ex}}(\mathbf{x}, t) = \sum_{\ell=\text{ex.}} \left[\hat{a}_\ell(t)u_\ell(\mathbf{x}) + \hat{a}_\ell^\dagger(t)v_\ell^*(\mathbf{x}) \right], \quad (10)$$

where “z.m.” and “ex.” represent summations over the zero and excitation modes, respectively. The operators

satisfy $[\hat{Q}_i(t), \hat{P}_j(t)] = i\delta_{ij}$, $[\hat{a}_\ell(t), \hat{a}_{\ell'}^\dagger(t)] = \delta_{\ell\ell'}$, and the vanishing ones otherwise, where $\hat{Q}_i(t)$ and $\hat{P}_i(t)$, also called the zero mode operators or quantum coordinates, are hermitian. Substituting the expansions (9) and (10) into Eq. (3), we obtain

$$\hat{H}_0 = \hat{H}_2 = \sum_{i=z.m.} \frac{\hat{P}_i^2}{2I_i^{-1}} + \sum_{\ell=ex.} \omega_\ell \hat{a}_\ell^\dagger \hat{a}_\ell. \quad (11)$$

The unperturbed Hamiltonian (11) is represented by the sum of the harmonic part of the excitation for each ℓ and the free part of the quantum coordinates for each i . Because of the latter contribution, we call this conventional formulation the free zero mode one. As a matter of course, there is no interaction among the zero modes in the Hamiltonian (11). The vacuum of the total system would be the ground state of Eq. (11): $|0\rangle = \prod_i |0\rangle_i \otimes |0\rangle_{ex}$, where $|0\rangle_i$ is the ground state in the i th zero mode sector satisfying the field division criterion $\langle 0|\hat{Q}_i|0\rangle_i = \langle 0|\hat{P}_i|0\rangle_i = 0$.

However, the choice of $|0\rangle_i$ causes problems. First, I_i may be negative, which corresponds to having a system with a negative “mass,” and there is no ground state. On the other hand, for $I_i > 0$, the lowest eigenstate is the zero “momentum” state, $\hat{P}_i|0\rangle_i = 0$. As a result of the uncertainty relation, the standard deviation of \hat{Q}_i , denoted by $\Delta Q_i = \sqrt{i\langle 0|\hat{Q}_i^2|0\rangle_i - i\langle 0|\hat{Q}_i|0\rangle_i^2}$, diverges. This divergence immediately conflicts with the starting assumption of small $\hat{\varphi}$. It also yields an unphysical situation in which some physical quantities such as the total number density also diverge, which will be seen in Eq. (29). One way to resolve this contradiction might be to choose a wave packet state as the vacuum [6]. Although ΔQ_i is finite at $t = 0$, it is again divergent after a long time because of the collapse of the wave packet; that is, $\hat{Q}_i(t) = \hat{Q}_i(0) + \hat{P}_i(0)t$ and $\Delta Q_i \propto t$ for large t . It is argued in Ref. [6] that \hat{Q}_i , acting as the phase operator of the order parameter, yields a divergent $\Delta Q(t)$ as t goes to infinity, and that the phase diffuses. All of the above pathological properties are rooted in the fact that the free Hamiltonian has a continuous spectrum.

We note that the difficulties concerning quantum fluctuations of zero modes are veiled in the Bogoliubov approximation, in which the original creation and annihilation operators associated with the eigenfunction belonging to zero eigenvalue are replaced with classical numbers in the field $\hat{\psi}$, or the zero mode operators in $\hat{\varphi}$ are simply neglected.

B. INTERACTING ZERO MODE FORMULATION

Although the choice of the unperturbed Hamiltonian (6) is based on the assumption of small $\hat{\varphi}$, or, more precisely, small $\hat{\varphi}_0$ and small $\hat{\varphi}_{ex}$, $\langle 0|\hat{Q}_i^2(t)|0\rangle$ is divergent, which indicates large $\hat{\varphi}_0$. Thus, the zero mode

fluctuations cannot be kept small, and the assumption of small $\hat{\varphi}_0$ has to be abandoned. This recognition is the starting point of Ref. [7]. Gathering all the terms consisting only of $\hat{\varphi}_0$ in the total Hamiltonian, we introduce the new unperturbed Hamiltonian

$$\hat{H}_u = \hat{H}_2 + \Delta\hat{H}, \quad (12)$$

where $\hat{H}_1 = 0$, for the same reason as in the previous subsection. The additional component, $\Delta\hat{H}$, is the sum of the third and fourth powers of the zero mode operators and their counter terms,

$$\Delta\hat{H} = \hat{H}_3^{QP} + \hat{H}_4^{QP} - \sum_{i=z.m.} [\delta\mu_i \hat{P}_i + \delta\nu_i \hat{Q}_i]. \quad (13)$$

The superscript QP indicates that all the terms consisting only of \hat{Q}_i and \hat{P}_i are picked up. We set up the stationary Schrödinger-like equation,

$$\hat{H}_u^{QP} |\Psi_\nu\rangle = E_\nu |\Psi_\nu\rangle \quad (\nu = 0, 1, 2, \dots). \quad (14)$$

As was seen in Ref. [7] for the model with the single zero mode and will be seen in the example with two zero modes in the next section, the eigenequation (14) is a type of bound state problem and yields a discrete spectrum, in contrast to the free zero mode formulation in the previous subsection. It is quite natural to take the whole unperturbed vacuum $|0\rangle = |\Psi_0\rangle \otimes |0\rangle_{ex}$, where $|\Psi_0\rangle$ is the ground state of the above equation. The unknown parameters $\delta\mu_i$ and $\delta\nu_i$ involved in \hat{H}_u^{QP} should be determined so as to satisfy the field division criterion

$$\langle \Psi_0|\hat{Q}_i|\Psi_0\rangle = \langle \Psi_0|\hat{P}_i|\Psi_0\rangle = 0 \quad (15)$$

in a manner consistent with Eq. (14).

Substituting the expansion (9) into Eq. (12), we gather it as $\hat{H}_u = \hat{H}_{u,1}^{QP} + \hat{H}_{u,2}^{QP} + \hat{H}_{u,3}^{QP} + \hat{H}_{u,4}^{QP} + \sum_\ell \omega_\ell \hat{a}_\ell^\dagger \hat{a}_\ell$, where

$$\hat{H}_{u,1}^{QP} = -\delta\mu_i \hat{P}_i - \delta\nu_i \hat{Q}_i, \quad (16)$$

$$\hat{H}_{u,2}^{QP} = \frac{\hat{P}_i^2}{2I_i^{-1}}, \quad (17)$$

$$\begin{aligned} \hat{H}_{u,3}^{QP} = 2\text{Re} \Big[& -iA_{\theta j k \ell} \hat{Q}_j \hat{Q}_k \hat{Q}_\ell + B_{\theta j k \ell} \hat{Q}_j \{\hat{Q}_k, \hat{P}_\ell\} \\ & - B_{k j \theta \ell}^* \hat{P}_\ell \hat{Q}_k \hat{Q}_j + iC_{\theta j k \ell} \hat{Q}_j \hat{P}_k \hat{P}_\ell \\ & - iC_{\theta j k \ell}' \hat{P}_k \{\hat{Q}_j, \hat{P}_\ell\} + D_{\theta j k \ell} \hat{P}_j \hat{P}_k \hat{P}_\ell \Big], \end{aligned} \quad (18)$$

$$\begin{aligned} \hat{H}_{u,4}^{QP} = & \frac{A_{ijk\ell}}{2} \hat{Q}_i \hat{Q}_j \hat{Q}_k \hat{Q}_\ell - \text{Im}[B_{ijk\ell}] \hat{Q}_i \hat{Q}_j \{\hat{Q}_k, \hat{P}_\ell\} \\ & - \text{Re}[C_{ijk\ell}] \hat{Q}_i \hat{Q}_j \hat{P}_k \hat{P}_\ell + \frac{C_{ijk\ell}'}{2} \{\hat{Q}_i, \hat{P}_k\} \{\hat{Q}_j, \hat{P}_\ell\} \\ & - \text{Im}[D_{ijk\ell}] \{\hat{Q}_i, \hat{P}_j\} \hat{P}_k \hat{P}_\ell + \frac{E_{ijk\ell}}{2} \hat{P}_i \hat{P}_j \hat{P}_k \hat{P}_\ell, \end{aligned} \quad (19)$$

with

$$\begin{aligned}
A_{ijkl} &= g \int dx f_i^* f_j^* f_k f_\ell, \quad B_{ijkl} = g \int dx f_i^* f_j^* f_k h_\ell, \\
C_{ijkl} &= g \int dx f_i^* f_j^* h_k h_\ell, \quad C'_{ijkl} = g \int dx f_i^* f_j h_k^* h_\ell, \\
D_{ijkl} &= g \int dx f_i^* h_j^* h_k h_\ell, \quad E_{ijkl} = g \int dx h_i^* h_j^* h_k h_\ell.
\end{aligned} \tag{20}$$

In Eqs. (18) and (19), we define $\{\hat{O}_1, \hat{O}_2\} \equiv \hat{O}_1 \hat{O}_2 + \hat{O}_2 \hat{O}_1$, and the dummy indices should be summed over. A remarkable consequence of the present formulation is that we have cross-terms among the different zero mode operators in the unperturbed Hamiltonian, namely, interactions and mixings among them. In other words, the Hamiltonian \hat{H}_u is an effective Hamiltonian in the zero mode sector governing the dynamics of the condensate and is uniquely derived from the original Hamiltonian (1).

III. APPLICATION TO HOMOGENEOUS SYSTEM WITH DARK SOLITON

In this section, we consider a system consisting of a dark soliton in a homogeneous system, which is described by the Hamiltonian in Eq. (1) with $V_{\text{ex}} = 0$ and has two zero modes. The GP solution of a one-dimensional dark soliton is

$$\xi(x) = \sqrt{n_0} \tanh\{\kappa(x - x_0)\}, \quad \mu = gn_0, \tag{21}$$

where $\kappa = \sqrt{mgn_0}$, and n_0 is the bulk density of the condensate. Hereafter, we set $x_0 = 0$ and $n_0 = 1$ for the sake of simplicity. The zero modes and their adjoint eigenfunctions are

$$f_\theta = \xi(x), \tag{22}$$

$$f_x = i \frac{d}{dx} \xi(x), \tag{23}$$

$$h_\theta = \frac{\sqrt{g}}{2L} [\tanh(\kappa x) + \kappa x \{1 - \tanh^2(\kappa x)\}], \tag{24}$$

$$h_x = -\frac{i}{4}, \tag{25}$$

$$I_\theta = \frac{g}{L}, \quad I_x = -\frac{g}{4\kappa}, \tag{26}$$

where the subscripts θ and x denote the $U(1)$ gauge and translational modes, respectively. We plot the four functions for later convenience in Fig. 1. Note that although in Ref. [8], which considered a nonperturbative treatment of the two zero modes, the system is put in an artificial box with a size L and the boundary condition $f_\theta(-L/2) = f_\theta(L/2) = 0$, our boundary condition is antiperiodic, $f_\theta(-L/2) = -f_\theta(L/2) \neq 0$. The operators \hat{Q}_θ and \hat{Q}_x , which are associated with the eigenfunctions

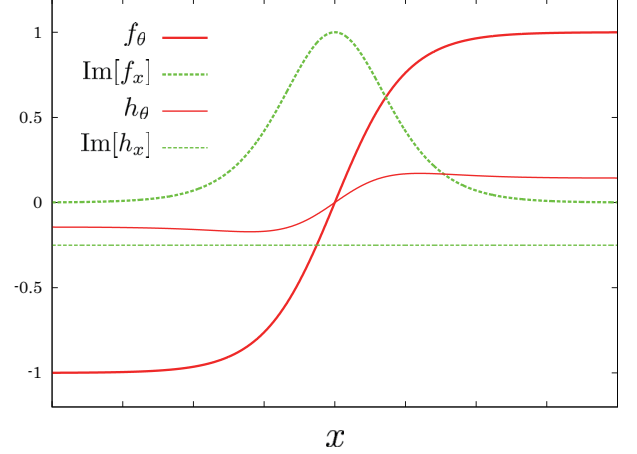


FIG. 1. (Color online) Plots of the zero modes and their adjoint eigenfunctions. Red bold and thin solid lines indicate f_θ and h_θ , respectively. Green bold and thin broken lines indicate f_x and h_x , respectively.

(22) and (23), respectively, may be interpreted as the phase and position operators of the soliton,

$$\begin{aligned}
\hat{\psi}(x) &= \xi(x) - i\hat{Q}_\theta \xi(x) + \hat{Q}_x \frac{d\xi(x)}{dx} + \dots \\
&\simeq \xi(x + \hat{Q}_x) e^{-i\hat{Q}_\theta}.
\end{aligned} \tag{27}$$

However, this interpretation is true only when \hat{Q}_θ and \hat{Q}_x are small. Our approach does not rely on it.

A. FREE ZERO MODE APPROACH FOR DARK SOLITON

In the conventional treatment, the unperturbed Hamiltonian (11) for this system is described by

$$\hat{H}_0 = \frac{\hat{P}_\theta^2}{2I_\theta^{-1}} + \frac{\hat{P}_x^2}{2I_x^{-1}} + \sum_{\ell=\text{ex.}} \omega_\ell \hat{a}_\ell^\dagger \hat{a}_\ell, \tag{28}$$

with positive I_θ and negative I_x . As mentioned above, the ground state for the $U(1)$ gauge zero mode induces phase diffusion, and there is no ground state for the translational zero mode because of the negative “mass,” $I_x < 0$. In Ref. [8], Eq. (28) is derived from the classical Lagrangian for the collective coordinates as the starting point of the nonperturbative treatment, and the $U(1)$ gauge zero mode sector is restricted to a subspace with a definite total number of atoms, assuming a large phase fluctuation. After this procedure, the calculated ΔQ_x with a Gaussian wave packet state grows with t . As can

be seen from the total number density

$$\begin{aligned} \langle 0 | \hat{\psi}^\dagger(x, t) \hat{\psi}(x, t) | 0 \rangle \\ = [1 + \Delta Q_\theta^2(t)] |\xi(x)|^2 + \Delta Q_x^2(t) \left| \frac{d\xi(x)}{dx} \right|^2 + \dots, \end{aligned} \quad (29)$$

ΔQ_x is related to the number of atoms that fill the center of the soliton. It has been indicated that $\Delta Q_x(t)$ grows with t , which is quantum depletion. A similar result of quantum depletion for a system with an attractive interaction having a bright soliton was obtained in Ref. [14], although then the “mass” I_x is positive.

The difficulty of the free zero mode approach lies in the the negative “mass” problem, the absence of a ground state for the unperturbed Hamiltonian (28), and the fact that $\Delta Q_x(t)$ increases to infinity with time.

B. INTERACTING ZERO MODE APPROACH FOR DARK SOLITON

In the interacting zero mode approach, we can obtain the natural vacuum, which causes neither phase dissipation nor the negative “mass” problem, by solving the Schrödinger-like equation (14) with the unperturbed Hamiltonian (12).

First, one has the nonperturbative Hamiltonian consisting only of \hat{Q}_i and \hat{P}_i with i equal to either θ or x , denoted by $\hat{H}_{u,i}^{QP}$ ($i = \theta, x$). In addition, there are interactions between $\{\hat{Q}_\theta, \hat{P}_\theta\}$ and $\{\hat{Q}_x, \hat{P}_x\}$, on which we focus our attention. We seek the dominant contribution of the interaction terms in the present model. Equations (22)–(25) and Fig. 1 show that f_θ and h_θ are odd functions with respect to the variable x , whereas f_x and h_x are even ones. The indices (i, j, k, ℓ) of the non-vanishing cross-term coefficients in Eq. (20) must be (θ, θ, x, x) in random order. We consider the limit of large L , much larger than the coherent length. Then the magnitude of h_x is small, *i.e.*, $1/L$, so the D and E terms can be neglected. The function f_x peaks sharply around $x = 0$, and the contributions of the A and B terms are small. Thus, the dominant contributions come only from the C and C' terms. One can neglect the third-power terms with respect to the zero mode operators because their contributions are small compared with those of the fourth-power terms owing to the field division criterion. Consequently, we have the approximate Hamiltonian

$$\hat{H}_u^{QP} \simeq \hat{H}_{u,\theta}^{QP} + \hat{H}_{u,x}^{QP} + 3|C_{\theta\theta xx}| \hat{Q}_\theta^2 \hat{P}_x^2. \quad (30)$$

Here the last term represents the dominant interaction between the U(1) gauge and translational zero modes. Solving the Schrödinger-like equation (14) with the Hamiltonian (30) numerically, we find the ground state or vacuum in the zero mode sector. To illustrate the significance of the mutual interaction, namely, the last term in Eq. (30), we plot the ground state distribution $|\Psi_0(Q_\theta, Q_x = 0)|^2$ with and without it in Fig. 2; the

difference between the plots is striking. The sharp distribution in the presence of the mutual interaction can be understood from the fact that the last term, behaving as Q_θ^2 , serves as an additional harmonic potential for Q_θ .

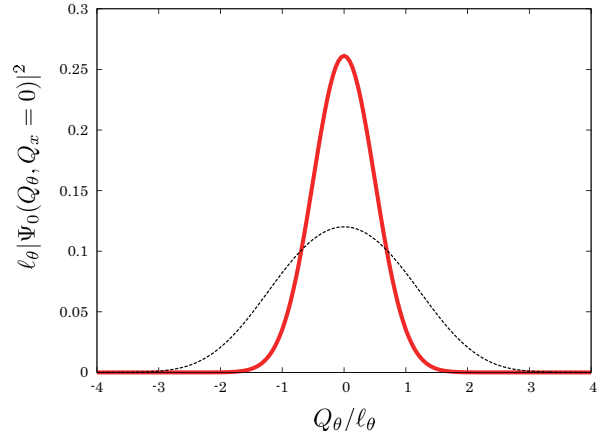


FIG. 2. (Color online) Ground state distribution $|\Psi_0(Q_\theta, Q_x = 0)|^2$ with and without the mutual interaction for $g = 1$ and $L = 1000$. (Red) bold solid and thin broken lines indicate the distributions for \hat{H}_u^{QP} and $\hat{H}_u^{QP} + \hat{H}_{u,\theta}^{QP}$, respectively. Axes are scaled by $\ell_\theta = \sqrt[6]{I_\theta/24A_{\theta\theta\theta\theta}}$ (see Ref. [7]).

One can evaluate the standard deviations ΔQ_i using the ground state distribution obtained above. The results versus the parameter g are presented in Fig. 3, which shows that ΔQ_θ and ΔQ_x decrease according to the power law $g^{-\alpha_i}$, with $\alpha_\theta = 0.092 \dots$ and $\alpha_x = -0.192 \dots$. It is natural that ΔQ_x decreases as g increases, as the coherent length is proportional to $g^{-1/2}$ and the soliton becomes sharper for larger g . We emphasize that the mutual interaction between the two zero modes qualitatively changes the g dependence of ΔQ_θ : There would be no g dependence without the mutual interaction, as in a normal condensate [7]. The suppression of ΔQ_θ because of the mutual interaction may be rephrased as the condensate phase becoming rigid because of the presence of a soliton with a π phase kink.

So far we have considered a homogeneous limit, *i.e.*, the large L limit. We turn our attention to finite- L systems and estimate the L dependence, for L is a controllable parameter in experiments, although in that case the translational symmetry is broken not spontaneously but explicitly. The calculations of the interacting zero mode formulation presented above are straightforwardly applied to a cylindrical system with a circumferential length $2L$ and two dark solitons, or to a system consisting of a dark soliton confined in a box of size L . For the latter, although our antiperiodic boundary condition is not realistic, we expect that the calculations reflect the effects of finite L qualitatively. The L dependence of ΔQ_i

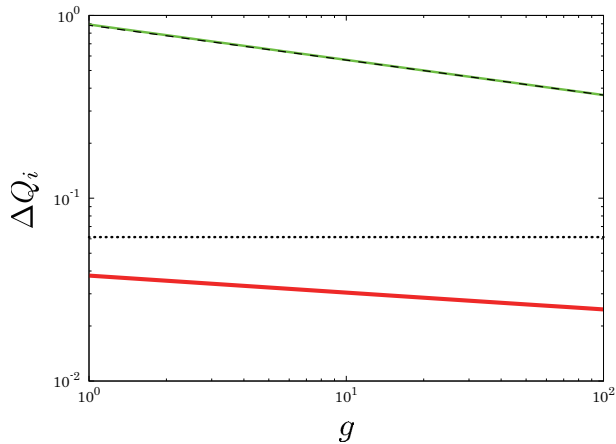


FIG. 3. (Color online) g dependences of standard deviations ΔQ_θ and ΔQ_x with $m = 1$ and $N_0 = 10^3$. (Red) bold and (green) thin solid lines indicate ΔQ_θ and ΔQ_x for the ground state of \hat{H}_u^{QP} in Eq. (30), respectively. Broken and dotted lines indicate the same quantities for the ground state of the Hamiltonian $\hat{H}_{u,\theta}^{QP} + \hat{H}_{u,x}^{QP}$, namely, for the ground state without the mutual interaction between the two zero modes.

is shown in Fig. 4, which shows that ΔQ_θ decreases as L^{β_θ} with $\beta_\theta = -0.440\dots$, whereas ΔQ_x increases as L^{β_x} with $\beta_x = 0.115\dots$. It is not surprising that ΔQ_θ de-

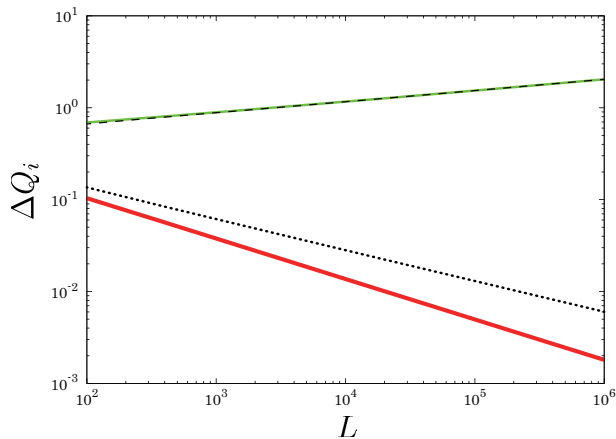


FIG. 4. (Color online) Effect of system size on standard deviations ΔQ_θ and ΔQ_x with $m = 1$ and $g = 1$. Each line indicates the same quantity as in Fig. 3.

pend on L , as the eigenfunction $f_\theta(x)$ in Eq. (22) extends over the entire range from $-L/2$ to $L/2$. On the other hand, the fact that ΔQ_x also clearly depends on L is puzzling at first glance, because the eigenfunction $f_x(x)$ in Eq. (23), with which the position operator of the soli-

ton \hat{Q}_x is associated, has a sharp distribution around the center of the soliton and is not affected by L . The puzzle can be resolved as follows: Whereas \hat{Q}_x describes the position of the soliton, its conjugate partner \hat{P}_x corresponds to its momentum or velocity and is size-dependent, as the spread of $h_x(x)$ is of order L [see Eqs. (9) and (25)]. It is well known that although a standing soliton has a π phase kink, a moving one has a smaller kink [15]. In view of this, the global character of $h_x(x)$, as seen in Eq. (25) to untwist the phase kink, could be understood. Because \hat{Q}_x and \hat{P}_x are canonically conjugate to each other, and one has the uncertainty relation $\Delta Q_x \cdot \Delta P_x \sim 1/2$, the standard deviation ΔQ_x depends on L as a consequence of the L dependence of ΔP_x . The size effects may be observed in experiments on finite-size systems in which L is controlled.

IV. SUMMARY

Considering that the zero mode is the essence of SSB and that its quantum fluctuation must be treated properly, we adopted the interacting zero mode formulation and extended it from a single zero mode system to a general case of multiple zero modes. It yields the effective Hamiltonian of a pair of canonical operators for each zero mode, the spectrum of which is discrete, as in the case of a single zero mode system, and introduces interactions among the zero modes naturally and definitely. The physical picture of zero modes interacting with each other is quite new.

As an application of the new formulation, a system of size L with a dark soliton is considered. In the large L or homogeneous limit, there are two zero modes corresponding to spontaneous breakdown of the $U(1)$ gauge and translational symmetries. We investigated this system by performing calculations. The vacuum is obtained uniquely, and the standard deviations for the zero mode operators ΔQ_i can be evaluated. The mutual interaction between the two zero modes influences the ground state distribution and therefore ΔQ_θ , and its effect is seen in the way that ΔQ_i depends on the coupling constant g .

As the trapped ultracold atomic system has a finite size L in real experimental situations, we also studied the L dependence of ΔQ_i , keeping L finite in our calculations. The results may be checked in experiments on a cylindrical system with two dark solitons or a dark soliton system confined in a finite region.

As a characteristic of soliton physics, the behavior of the translational zero mode of the dark soliton is expected to correlate with its velocity. A study of the correlation is a future work.

ACKNOWLEDGMENTS

This work is supported in part by a Grant-in-Aid for Scientific Research (C) (No. 25400410) from the Japan

Society for the Promotion of Science, Japan; “Ambient SoC Global Program of Waseda University” of the Min-

istry of Education, Culture, Sports, Science and Technology, Japan; and Waseda University Grant for Special Research Projects (Project No. 2013B-102).

-
- [1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, *Science* **269**, 198 (1995).
 - [2] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, *Phys. Rev. Lett.* **75**, 3969 (1995).
 - [3] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, *Phys. Rev. Lett.* **75**, 1687 (1995).
 - [4] J. Goldstone, *Nuovo Cimento* **19**, 154 (1961).
 - [5] Y. Nambu and G. Yona-Lasinio, *Phys. Rev.* **122**, 345 (1961).
 - [6] M. Lewenstein and L. You, *Phys. Rev. Lett.* **77**, 3489 (1996).
 - [7] Y. Nakamura, J. Takahashi, and Y. Yamanaka, *Phys. Rev. A* **89**, 013613 (2014).
 - [8] J. Dziarmaga, *Phys. Rev. A* **70**, 063616 (2004).
 - [9] H. Matsumoto and S. Sakamoto, *Prog. Theor. Phys.* **107**, 679 (2002).
 - [10] M. Mine, M. Okumura, and Y. Yamanaka, *J. Math. Phys.* **46**, 042307 (2005).
 - [11] E. P. Gross, *Nuovo Cimento* **20**, 454 (1961); *J. Math. Phys.* **4**, 195 (1963). L. P. Pitaevskii, *Zh. Eksp. Teor. Fiz. [Sov. Phys. JETP]* **40**, 546 (1961); *Sov. Phys. JETP* **13**, 451 (1961).
 - [12] N. N. Bogoliubov, *J. Phys. (Moscow)* **11**, 32 (1947).
 - [13] P. G. de Gennes, *Superconductivity of Metals and Alloys* (Benjamin, New York, 1966).
 - [14] G. Huang, L. Deng, J. Yan, and B. Hu, *Phys. Lett. A* **357**, 150 (2006).
 - [15] D. J. Frantzeskakis, *J. Phys. A: Math. Theor.* **43**, 213001 (2010).